



Topology: A Series of Exercises and Solutions

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Knowledge in book, insight in mind; NOTE IN ANOTHER FILE

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Part I

Armstrong's Exercises

Chapter 1 1

Chapter 2 2

Chapter 3 Compactness and Connectedness

✚ **Exercise 3.6** Show that the Hausdorff condition cannot be relaxed in the following theorem: a one-one, onto, and continuous function from a compact space X to a Hausdorff space Y is a homeomorphism.

Proof Consider a compact space $X = \{1, 2, 3, 4\}$ with discrete topology and a Hausdorff space $Y = \{1, 2, 3, 4\}$ with topology $\{\emptyset, \{1, 2\}, \{3, 4\}, Y\}$. Then the identity function $f : X \rightarrow Y$ is one-one, onto, and continuous. However, f is not a homeomorphism since $f(\{1\}) = \{1\}$ is not open in Y .

✚ **Exercise 3.8** If X has a countable base for its topology, prove that any open cover of X contains a countable subcover.

Proof Let $\mathcal{B} = \{B_n\}_{n=1}^\infty$ be a countable (infinite, more generally speaking) base for the topology of X . With the axiom of choice, we consider function when given an arbitrary nonempty open cover $\{U_\alpha\}_{\alpha \in \Lambda}$

$$f : \mathbb{N} \rightarrow \Lambda, \quad n \mapsto \begin{cases} \lambda', & \text{if no members of } \{U_\alpha\} \text{ contain } B_n \\ \lambda, & \text{if } U_\lambda \supset B_n \end{cases},$$

where $U_{f(n)}$ is one of subsets containing B_n , and λ' is a fixed element of Λ . Then $\{U_{f(n)}\}_{n=1}^\infty$ is a countable subcover of $\{U_\alpha\}_{\alpha \in \Lambda}$. Otherwise, there exists $x \in X$ not in any $U_{f(n)}$, which means $x \in U_\alpha$ for some $\alpha \in \Lambda \setminus f(\mathbb{N})$. Since $\mathcal{B}$ is a base, there exists $B_{n_0} \in \mathcal{B}$ such that $x \in B_{n_0} \subset U_\alpha$. Then $U_{f(n_0)} \supset B_{n_0}$, which implies that $x \in U_{f(n_0)}$, a contradiction.

✚ **Exercise 3.9** Prove that two disjoint compact subsets of a Hausdorff space always possess disjoint neighbourhoods.

Proof For any $x \in A, y \in B$, there exist disjoint neighbourhoods $U_{(x,y)}, V_{(x,y)}$ of x, y respectively. Then $\{V_{(x,y)}\}_{y \in B}$ is an open cover of B for a given $x \in A$. Since B is compact, there exists a finite subcover $\{V_{(x,y_i,x)}\}_{i=1}^{n_x}$ of B . Then $\bigcap_{i=1}^{n_x} U_{(x,y_i,x)}$ is a neighbourhood of x disjoint from $\bigcup_{i=1}^{n_x} V_{(x,y_i,x)}$, a neighbourhood of B . Since A is compact, there exists a finite subcover $\{\bigcap_{i=1}^{n_{x_j}} U_{(x_j,y_i,x_j)}\}_{j=1}^m$ of A . Then $\bigcup_{j=1}^m \bigcap_{i=1}^{n_{x_j}} U_{(x_j,y_i,x_j)}$ is a neighbourhood of A disjoint from $\bigcap_{j=1}^m \bigcup_{i=1}^{n_{x_j}} V_{(x_j,y_i,x_j)}$, a neighbourhood of B .

✚ **Exercise 3.10** Let A be a compact subset of a metric space X . Show that the diameter of A is equal to $d(x, y)$ for some pair of points $x, y \in A$. Given $x \in X$, show that $d(x, A) = d(x, y)$ for some $y \in A$. Given a closed subset B , disjoint from A , show that $d(A, B) > 0$.

Proof By the definition of diameter, there exists a sequence $\{(x_n, y_n)\}_{n=1}^\infty \subset A \times A$ such that $d(x_n, y_n) \rightarrow \text{diam}(A)$. Since A is compact in a metric space, it is sequentially compact. Then there exists a limit $(x, y) \in A \times A$ of mentioned sequence, satisfying $d(x, y) = \text{diam}(A)$. For any $x \in X$, consider the function $f : A \rightarrow \mathbb{R}$ defined by $f(y) = d(x, y)$. Since f is continuous and A is compact, there exists $y \in A$ such that $d(x, A) = d(x, y)$, reaching the minimum of f on A . For any closed subset B disjoint from A , if $d(A, B) = 0$, then there exists a sequence $\{(x_n, y_n)\}_{n=1}^\infty \subset A \times B$ such that $d(x_n, y_n) \rightarrow 0$. Since A is compact, there exists a limit point $x \in A$ of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$. Then $d(x, y_{n_k}) \rightarrow 0$, which implies that $y_{n_k} \rightarrow x \in B$ since B is closed. This is a contradiction because $A \cap B = \emptyset$.

✚ **Exercise 3.11** Find a topological space and a compact subset whose closure is not compact.

Proof Let the topology of $\mathbb{N}$ be generated by $\{\{1, n\} : n \geq 2\} \cup \{\emptyset, \mathbb{N}\}$. Then $\{1\}$ is compact but the closure $\mathbb{N}$ is not compact since $\{\{1, n\} : n \geq 2\}$ is an open cover of $\mathbb{N}$ with no finite subcover.

✚ **Exercise 3.13** Let $f : X \rightarrow Y$ be a closed map with the property that the inverse image of each point of Y is a compact subset of X . Show that $f^{-1}(K)$ is compact whenever K is compact in Y . Can you remove the condition that f be closed?

Proof To show $f^{-1}(K)$ is compact, it only needs to show that for any open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of $f^{-1}(K)$, there exists a finite subcover. For any $y \in K$, $f^{-1}(y)$ is compact, so there exists a finite subcover $\{U_{\alpha(y,n)}\}_{n=1}^{N_y}$ of $f^{-1}(y)$, the union of whose members is an open neighbourhood of $f^{-1}(y)$, denoted by V_y . Note that

$$f^{-1}(y) \subset f^{-1}(f(V_y)^c) \subset V_y = \bigcup_{n=1}^{N_y} U_{\alpha(y,n)},$$

where $f(V_y)^c$ is open since f is closed. Then $\{f(V_y)^c\}_{y \in K}$ is an open cover of K , which has a finite subcover $\{f(V_{y_m})^c\}_{m=1}^M$. Then $\{U_{\alpha(y_m,n)}\}_{m=1, n=1}^{M, N_{y_m}}$ is a finite subcover of $f^{-1}(K)$. The condition that f be closed is necessary. Consider

$$f : \{(x, y) \in \mathbb{R}^2 : y = x^{-1}\} \rightarrow \mathbb{R}, \quad (x, y) \mapsto x,$$

which satisfies all the conditions except that it is not closed since the image of f is $\mathbb{R} \setminus \{0\}$. One takes that $K = [-1, 1]$

is compact, while $f^{-1}(K)$ is not compact because of unboundedness.

✚ **Exercise 3.14** If $f : X \rightarrow Y$ is a one-one map, and if $f : X \rightarrow f(X)$ is a homeomorphism when we give $f(X)$ the induced topology from Y , we call f an embedding of X in Y . Show that a one-one map from a compact space to a Hausdorff space must be an embedding.

Proof It remains to show that $f^{-1} : f(X) \rightarrow X$ is continuous, equivalently, that f is closed. Since f is continuous, X is compact and $f(X)$ is Hausdorff, for any closed subset $A \subset X$, $f(A)$ is compact and thus closed in $f(X)$.

✚ **Exercise 3.21** If A and B are compact, and if W is a neighbourhood of $A \times B$ in $X \times Y$, find a neighbourhood U of A in X and a neighbourhood V of B in Y such that $U \times V \subset W$.

Proof For any $x \in A$, $\{x\} \times B$ is compact. Denote $U_{(x,y)}$ as a base neighbourhood (can be written as the product of two open sets of x and y respectively) of $(x, y) \in A \times B$ contained in W . Then there exists a finite cover $\{U_{(x,y_n)}\}_{n=1}^N$ on $\{x\} \times B$, so

$$\{x\} \times B \subset \bigcap_{n=1}^{N_x} \pi_X[U_{(x,y_{n,x})}] \times \bigcup_{n=1}^{N_x} \pi_Y[U_{(x,y_{n,x})}] \subset W.$$

Note that $A \times B$ is compact, so there also exists a finite cover

$$A \times B \subset \bigcup_{m=1}^M \left(\bigcap_{n=1}^{N_{x_m}} \pi_X[U_{(x_m,y_{n,x_m})}] \times \bigcup_{n=1}^{N_{x_m}} \pi_Y[U_{(x_m,y_{n,x_m})}] \right) \subset W.$$

More specifically, we obtain a neighbourhood U of A in X and a neighbourhood V of B in Y such that

$$A \times B \subset \left(\bigcup_{m=1}^M \bigcap_{n=1}^{N_{x_m}} \pi_X[U_{(x_m,y_{n,x_m})}] \right) \times \left(\bigcap_{m=1}^M \bigcup_{n=1}^{N_{x_m}} \pi_Y[U_{(x_m,y_{n,x_m})}] \right) \subset W.$$

✚ **Exercise 3.25** Show that the diagonal map $\Delta : X \rightarrow X \times X$ defined by $\Delta(x) = (x, x)$ is indeed a map, and check that X is Hausdorff iff $\Delta(X)$ is closed in $X \times X$.

Proof For any basic open set $U \times V$ in $X \times X$, where U, V are open in X , it holds

$$\Delta^{-1}(U \times V) = \{x \in X : (x, x) \in U \times V\} = \{x \in X : x \in U \cap V\} = U \cap V,$$

which is open in X . Therefore, Δ is a map. If X is Hausdorff, then for any $(x, y) \in (X \times X) \setminus \Delta(X)$, there exist disjoint neighbourhoods U, V of x, y respectively. Then $U \times V$ is a neighbourhood of (x, y) not intersecting $\Delta(X)$, which implies that $\Delta(X)$ is closed. The converse is analogous. For any $(x, y) \in (X \times X) \setminus \Delta(X)$, there exists a neighbourhood (choosing basic open sets) $U \times V$ of (x, y) not intersecting $\Delta(X)$. Then U, V are disjoint neighbourhoods of x, y respectively.

✚ **Exercise 3.26** We know that the projections $p_1 : X \times Y \rightarrow X, p_2 : X \times Y \rightarrow Y$ are open maps. Are they always closed?

Proof Not always. Consider $X = Y = \mathbb{R}$ and $A = \{(x, y) : xy = 1\}$, which is closed in $\mathbb{R}^2$. However, $p_1(A) = \mathbb{R} \setminus \{0\}$ is not closed in $\mathbb{R}$.

Part II

Complementary Exercises

 **Exercise 3.27** In the class, we proved that a subset in $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

- (1) Give an example to show that a closed and bounded subset in a metric space is not necessarily compact in general.
- (2) Let X be a metric space and let $\varepsilon > 0$. A subset $E \subset X$ is called an ε -grid if for any $x \in X$ there exists $e \in E$ such that $d(e, x) < \varepsilon$. A metric space is called totally bounded if for any $\varepsilon > 0$ there exists a finite ε -grid. Prove that a subset in a complete metric space is compact if and only if it is closed and totally bounded.

Proof We will prove the two statements in the exercise separately.

- (1) In an incomplete metric space, a closed and bounded subset need not be compact. Take $(\mathbb{R}' := \mathbb{R} \setminus \{0\}; d)$ as an example, which is a subspace of $(\mathbb{R}; d)$. The subset $(0, 1] = [0, 1] \cap \mathbb{R}'$ is closed and bounded in $\mathbb{R}'$, but it is not compact since the open cover $\{(1/n, 2) : n \in \mathbb{N}\}$ has no finite subcover.
- (2) Let E be a subset in a complete metric space X . If E is compact, then it is closed since X is Hausdorff space. For any $\varepsilon > 0$, consider $\bigcup_{x \in E} B(x, \varepsilon)$, which is an open cover of E and has a finite subcover due to the compactness of E . The centers of the balls in the finite subcover form a finite ε -grid of E , which shows that E is totally bounded. Conversely, for any $n \in \mathbb{N}$, there exists a finite $1/n$ -grid E_n of E , forming an open cover $\{B(x, 1/n) : x \in E_n\}$ of E . Then for any open cover $\{U_\alpha\}_{\alpha \in \Lambda}$ of E in X , one can claim that there exists $N \in \mathbb{N}$ such that for any $x \in E_N$, $B(x, 1/N)$ is contained in some U_α . Otherwise, for any $n \in \mathbb{N}$, there exists $x_n \in E_n$ such that $B(x_n, 1/n)$ is not contained in any U_α . Since E is (totally bounded) bounded and closed, there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with limit $x \in E$. Then for any U_α containing x , there exists $N' \in \mathbb{N}$ such that $B(x, 1/N') \subset U_\alpha$. For any k large enough such that $n_k > 2N'$, we have $d(x_{n_k}, x) < 1/2N'$, which implies $B(x_{n_k}, 1/n_k) \subset B(x, 1/N') \subset U_\alpha$, a contradiction. Therefore, there exists $N \in \mathbb{N}$ such that for any $x \in E_N$, $B(x, 1/N)$ is contained in some U_α . Since E_N is finite set, we find a finite subcover of $\{U_\alpha\}$, implying the compactness of E .

✚ **Exercise 3.20** If $X \times Y$ has the product topology, and if $A \subset X, B \subset Y$, show that $\overline{A \times B} = \overline{A} \times \overline{B}$, $(A \times B)^\circ = \mathring{A} \times \mathring{B}$, and $\partial(A \times B) = (\partial A \times \overline{B}) \cup (\overline{A} \times \partial B)$.

Proof For any $(x, y) \in \overline{A} \times \overline{B}$, x, y are limit points of A, B respectively. That is, there are sequences $x_n \rightarrow x, y_n \rightarrow y$ with $x_n \in A, y_n \in B$ for all $n \in \mathbb{N}$. So (x, y) is the limit of the sequence $(x_n, y_n) \in A \times B$.

Note that $\mathring{A} \times \mathring{B}$ is open and $A \times B \supset \mathring{A} \times \mathring{B}$. By taking the interior again one obtains $(A \times B)^\circ \supset \mathring{A} \times \mathring{B}$. Conversely, for any $(x, y) \in (A \times B)^\circ$, there is a neighbourhood $U_{(x,y)} \subset A \times B$. According to the product space, the projection π_X is open, so that $x \in \pi_X(U_{(x,y)}) \subset \pi_X(A \times B) = A$, which means $x \in \mathring{A}$. Analogously $y \in \mathring{B}$.

With previous result, one can notice that

$$\partial(A \times B) = \overline{A \times B} \cap \overline{(A \times B)^c} = (\overline{A} \times \overline{B}) \cap ((A \times B)^\circ)^c = (\overline{A} \times \overline{B}) \cap (\mathring{A} \times \mathring{B})^c$$

Chapter 3 Compactness and Connectedness

✚ **Exercise 3.14** hi

✚ **Exercise 3.30** ★ Let X be the set of all points in the plane which have at least one rational coordinate. Show that X , with the induced topology, is a connected space.

Proof Points in X can be connected to at least one axis by a parallel line. Since the axes are connected, X is connected.

✚ **Exercise 3.31** Give the set of real numbers the finite-complement topology. What are the components of the resulting space? Answer the same question for the half-open interval topology.

Proof

✚ **Exercise 3.32** ★ If X has only a finite number of components, show that each component is both open and closed. Find a space none of whose components are open sets.

Proof It's known that components are closed, so is the union of finitely many components. Therefore, each component is open. However, the finiteness of the number of components is necessary. Consider the space $X = \mathbb{Q}$ with the subspace topology from $\mathbb{R}$. Then the components of X are singletons, which are not open in X .

✚ **Exercise 3.33** ★ (*Intermediate value theorem*). If $f : [a, b] \rightarrow \mathbb{E}^1$ is a map such that $f(a) < 0$ and $f(b) > 0$, use the connectedness of $[a, b]$ to establish the existence of a point c for which $f(c) = 0$.

Proof It's known that the continuous image of a connected space is connected. So $0 \in f([a, b])$. Otherwise, $f([a, b])$ can be covered by two disjoint open sets $(-\infty, 0)$ and $(0, +\infty)$, which contradicts the connectedness of $f([a, b])$.

✚ **Exercise 3.34** ★ A space X is *locally connected* if for each $x \in X$, and each neighbourhood U of x , there is a connected neighbourhood V of x which is contained in U . Show that any euclidean space, and therefore any space which is locally euclidean (like a surface), is locally connected. If $X = \{0\} \cup \{1/n \mid n = 1, 2, \dots\}$ with the subspace topology from the real line, show that X is not locally connected.

Proof Note that any euclidean space $\mathbb{E}^n$ possesses a base of open balls $\{B(x, r) : x \in \mathbb{E}^n, r > 0\}$, which are all connected. Therefore, $\mathbb{E}^n$ is locally connected. The space $X = \{0\} \cup \{1/n : n \in \mathbb{N}\}$ is not locally connected, since for any neighbourhood U of 0, there exists a ball $B(0, r) \subset U$ for some $r > 0$ so that $U \subset X$ contains infinitely many points of the form $1/n$, which cannot be connected to 0 in X .

✚ **Exercise 3.35** ★★ Show that local connectedness is preserved by a homeomorphism, but need not be preserved by a continuous function.

Proof Let $f : X \rightarrow Y$ be a homeomorphism, where X is locally connected. Then for any neighbourhood U of $f(x)$ in Y , $f^{-1}(U)$ is a neighbourhood of x in X . Since X is locally connected, there exists a connected neighbourhood $V \subset f^{-1}(U)$ of x . Then $f(V)$ is a connected neighbourhood of $f(x)$ contained in U because f is continuous, preserving connectedness. Therefore, Y is locally connected. However, local connectedness need not be preserved by a continuous function. Consider the identity map $f : \mathbb{Q} \rightarrow \mathbb{Q}$ with the domain $\mathbb{Q}$ equipped with the discrete topology and the codomain $\mathbb{Q}$ equipped with the subspace topology from $\mathbb{R}$. Then f is continuous, but the domain is locally connected while the codomain is not.

✚ **Exercise 3.36** Show that X is locally connected iff every component of each open subset of X is an open set.

✚ **Exercise 3.37** ★ Show that the continuous image of a path-connected space is path-connected.

Proof Let $f : X \rightarrow Y$ be a map and X be path-connected. For any $y_1, y_2 \in f(X)$, there exist $x_1, x_2 \in X$ such that $f(x_1) = y_1, f(x_2) = y_2$. Since X is path-connected, there exists a path $\gamma : [0, 1] \rightarrow X$ such that $\gamma(0) = x_1, \gamma(1) = x_2$. Then $f \circ \gamma : [0, 1] \rightarrow Y$ is a path connecting y_1 and y_2 , which shows that $f(X)$ is path-connected.

✚ **Exercise 3.38** Show that S^n is path-connected for $n > 0$.

✚ **Exercise 3.39** ★ Prove that the product of two path-connected spaces is path-connected.

Proof Let X, Y be two path-connected spaces. For any $(x_1, y_1), (x_2, y_2) \in X \times Y$, there exist paths $\gamma_X : [0, 1] \rightarrow X$ and $\gamma_Y : [0, 1] \rightarrow Y$ such that $\gamma_X(0) = x_1, \gamma_X(1) = x_2$ and $\gamma_Y(0) = y_1, \gamma_Y(1) = y_2$. Then the map $\gamma : [0, 1] \rightarrow X \times Y$ defined by $\gamma(t) = (\gamma_X(t), \gamma_Y(t))$ is a path connecting (x_1, y_1) and (x_2, y_2) , which shows that $X \times Y$ is path-connected. By the way, one can check that γ is continuous since $\gamma^{-1}(U \times V) = \gamma_X^{-1}(U) \cap \gamma_Y^{-1}(V)$ is open in $[0, 1]$ for any basic open set $U \times V$ in $X \times Y$.

✚ **Exercise 3.40** ★ If A and B are path-connected subsets of a space, and if $A \cap B$ is nonempty, prove that $A \cup B$ is path-connected.

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- Proof** Choose a point $x_0 \in A \cap B$. Then for any $x \in A \cup B$, either $x \in A$ or $x \in B$. Since A, B are both path-connected, x_0, x are in the same path-connected component of $A \cup B$. Therefore, $A \cup B$ is path-connected.
-  **Exercise 3.41** Find a path-connected subset of a space whose closure is not path-connected.
-  **Exercise 3.42** Show that any indiscrete space is path-connected.
-  **Exercise 3.43** A space X is *locally path-connected* if for each $x \in X$, and each neighbourhood U of x , there is a path-connected neighbourhood V of x which is contained in U . Is the space shown in Fig. 3.4 locally path-connected? Convert the space $\{0\} \cup \{1/n \mid n = 1, 2, \dots\}$ into a subspace of the plane which is path-connected but not locally path-connected.
-  **Exercise 3.44** Prove that a space which is connected and locally path-connected is path-connected.

Chapter 4 Identification Spaces

✚ **Exercise 4.1** Check that the three descriptions (a), (b), (c) of P^n listed in ‘Projective spaces’ above do all lead to the same space.

✚ **Exercise 4.2** ★★? Which space do we obtain if we take a Möbius strip and identify its boundary circle to a point?

Proof $\mathbb{R}P^2$ is obtained. Consider the Möbius strip as a rectangle with two opposite sides identified in the opposite direction. Then the boundary circle of the Möbius strip is the union of the other two sides of the rectangle, which are identified to a point. Therefore, we obtain a triangle with three sides identified in pairs, which is homeomorphic to $\mathbb{R}P^2$.

✚ **Exercise 4.3** ★★★ Let $f : X \rightarrow Y$ be an identification map, let A be a subspace of X , and give $f(A)$ the induced topology from Y . Show that the restriction $f|_A : A \rightarrow f(A)$ need not be an identification map.

Proof Let $X = \mathbb{R}$ with the standard topology, and let $Y = S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. Define the map $f : X \rightarrow Y$ by $f(x) = e^{2\pi i x}$. Since f is a continuous open surjection, it is an identification map. Now, let

$$A = (0, 3/8) \cup (3/8, 3/4) \cup (3/2, 9/4)$$

be a subspace of X . The restriction $g = f|_A : A \rightarrow f(A)$ is obviously a continuous surjection. Note that $f(A) = S^1$. Consider the subset and its preimage

$$F = \{e^{2\pi i t} \mid t \in (2, 19/8]\} \subset S^1, \quad g^{-1}(F) = (0, 3/8) \cup (2, 9/4) \subset A.$$

$g^{-1}(F)$ is open in A but F is not open in S^1 . Therefore, g is not an identification map.

✚ **Exercise 4.4** ★ With the terminology of Exercise 4.3, show that if A is open in X and if f takes open sets to open sets, or if A is closed in X and f takes closed sets to closed sets, then $f|_A : A \rightarrow f(A)$ is an identification map.

Proof It is easy to check that $g = f|_A$ is continuous and surjective. If f takes open sets to open sets and A is open in X , then for any $U \subset f(A)$ satisfying $g^{-1}(U)$ is open in A , it holds that $g(g^{-1}(U)) = U \subset f(A)$ is open in $f(A)$, which means that g is an identification map. For the case of closedness, one can prove that every set $U \subset f(A)$ satisfying $g^{-1}(U)$ is closed in A is also closed in $f(A)$. Then for every set $U \subset f(A)$ satisfying $g^{-1}(U)$ is open in A , $g^{-1}(U)^c = g^{-1}(U^c)$ (since g is surjective) is open in A , which implies that U^c is closed in $f(A)$ and thus U is open in $f(A)$. Therefore, g is an identification map.

✚ **Exercise 4.5** ★ Let X denote the union of the circles $[x - (1/n)]^2 + y^2 = (1/n)^2$, $n = 1, 2, 3, \dots$, with the subspace topology from the plane, and let Y denote the identification space obtained from the real line by identifying all the integers to a single point. Show that X and Y are not homeomorphic. (X is called the *Hawaiian earring*.)

Proof Prove by contradiction. Suppose there exists a homeomorphism $f : X \rightarrow Y$. If $f(O)$ is not the identified point in Y , then consider $B(f(O), \delta)$ with $\delta > 0$ small enough such that $B(f(O), \delta)$ does not contain the identified point. Then $f^{-1}(B(f(O), \delta))$ is a neighbourhood of O in X , which contains infinitely many circles. However, the path

$$f^{-1} : [f(O) - \delta, f(O) + \delta] \rightarrow f^{-1}(B(f(O), \delta))$$

have to travel from one circle to another in $f^{-1}(B(f(O), \delta))$, travelling through O each time (more than once), which is a contradiction to the injectivity of f^{-1} . Therefore, $f(O)$ is the identified point in Y . Then consider $B(f(O), \delta)$ with $\delta > 0$ small enough such that $B(f(O), \delta)$ contains only the identified point and no other integers. So $B(f(O), \delta)^c$ contains infinitely many path-connected components, while $f^{-1}(B(f(O), \delta))$ is a neighbourhood of O in X , which contains $\{[x - (1/n)]^2 + y^2 = (1/n)^2\}$ for n larger than one certain integer N . Therefore, $f^{-1}(B(f(O), \delta)^c)$ contains no more than N path-connected components (since $f^{-1}(B(f(O), \delta))$ is open and path-connected), which is a contradiction to the continuity of f^{-1} . Therefore, X and Y are not homeomorphic.

✚ **Exercise 4.6** Give an example of an identification map which is neither open nor closed.

✚ **Exercise 4.7** Describe each of the following spaces: (a) the cylinder with each of its boundary circles identified to a point; (b) the torus with the subset consisting of a meridional and a longitudinal circle identified to a point; (c) S^2 with the equator identified to a point; (d) $\mathbb{E}^2$ with each of the circles centre the origin and of integer radius identified to a point.

✚ **Exercise 4.8** Let X be a compact Hausdorff space. Show that the cone on X is homeomorphic to the one-point compactification of $X \times [0, 1)$. If A is closed in X , show that X/A is homeomorphic to the one-point compactification of $X - A$.

✚ **Exercise 4.9** Let $f : X \rightarrow X'$ be a continuous function and suppose we have partitions $\mathcal{P}, \mathcal{P}'$ of X and X' respectively, such that if two points of X lie in the same member of $\mathcal{P}$, their images under f lie in the same member of $\mathcal{P}'$. If Y, Y' are the identification spaces given by these partitions, show that f induces a map $\hat{f} : Y \rightarrow Y'$, and that if f is an identification map then so is $\hat{f}$.

✚ **Exercise 4.10** ★ Let S^2 be the unit sphere in $\mathbb{E}^3$ and define $f : S^2 \rightarrow \mathbb{E}^4$ by $f(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Show that f induces an embedding of the projective plane in $\mathbb{E}^4$ (embeddings were defined in Exercise 3.14).

Proof Every component of values of f is continuous and $\text{Im } f \subset \mathbb{E}^4$, so f is continuous. For any $(x, y, z), (x', y', z') \in S^2$, if $f(x, y, z) = f(x', y', z')$, then

$$\begin{cases} x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2 = 1 \\ x^2 - y^2 = x'^2 - y'^2 \\ xy = x'y' \\ xz = x'z' \\ yz = y'z' \end{cases}.$$

If $x^2 = y'^2$, then $|x| = |y| = |x'| = |y'|$ and $|z| = |z'|$. If $x^2 \neq y'^2$, one can also obtain $|z| = |z'|$ by the second and last two equations. Therefore, it holds that

$$|x| = |x'|, |y| = |y'|, |z| = |z'|,$$

in all cases. As x, y, z are not all zero, one can solve $(x, y, z) = (x', y', z')$ or $(x, y, z) = (-x', -y', -z')$ by last three equations. So $(x, y, z) \sim_f (-x, -y, -z)$, which shows that f identifies antipodal points on S^2 . The induced map

$$\tilde{f} : S^2 / \sim_f \rightarrow \mathbb{E}^4, \quad [(x, y, z)] \mapsto f(x, y, z)$$

is well-defined and one-one. Since S^2 is compact (so is $S^2 / \sim_f$, since the identification map preserves compactness) and $\mathbb{E}^4$ is Hausdorff, $\tilde{f}$ is an embedding by the result of Exercise 3.14.

✚ **Exercise 4.11** ★ Show that the function $f : [0, 2\pi] \times [0, \pi] \rightarrow \mathbb{E}^5$ defined by

$$f(x, y) = (\cos x, \cos 2y, \sin 2y, \sin x \cos y, \sin x \sin y)$$

induces an embedding of the Klein bottle in $\mathbb{E}^5$.

Proof First, one can check that f is continuous and for any $(x, y), (x', y') \in [0, 2\pi] \times [0, \pi]$, if $f(x, y) = f(x', y')$, then

$$\begin{cases} \cos x = \cos x' \\ \cos 2y = \cos 2y' \\ \sin 2y = \sin 2y' \\ \sin x \cos y = \sin x' \cos y' \\ \sin x \sin y = \sin x' \sin y' \end{cases} \implies \begin{cases} x = x' \text{ or } x + x' = 2\pi \\ y = y' \\ \sin x \cos y = \sin x' \cos y' \\ \sin x \sin y = \sin x' \sin y' \end{cases}.$$

By checking the signs of last three equations, one can obtain that $(x, y) = (x', y')$, and $(x, y) = (2\pi - x', \pi - y')$ are the only two cases. Therefore, f identifies $(x, y) \sim_f (2\pi - x, \pi - y)$, which is the identification of the Klein bottle. The induced map

$$\tilde{f} : ([0, 2\pi] \times [0, \pi]) / \sim_f \rightarrow \mathbb{E}^5, \quad [(x, y)] \mapsto f(x, y)$$

is well-defined and one-one. Since $[0, 2\pi] \times [0, \pi]$ is compact (so is $([0, 2\pi] \times [0, \pi]) / \sim_f$) and $\mathbb{E}^5$ is Hausdorff, $\tilde{f}$ is an embedding by the result of Exercise 3.14. The statement is analogous to the proof of the Exercise 4.10.

✚ **Exercise 4.12** ★ With the notation of Exercise 4.11, show that if $(2 + \cos x) \cos 2y = (2 + \cos x') \cos 2y'$ and $(2 + \cos x) \sin 2y = (2 + \cos x') \sin 2y'$, then $\cos x = \cos x'$, $\cos 2y = \cos 2y'$, and $\sin 2y = \sin 2y'$. Deduce that the function $g : [0, 2\pi] \times [0, \pi] \rightarrow \mathbb{E}^4$ given by $g(x, y) = ((2 + \cos x) \cos 2y, (2 + \cos x) \sin 2y, \sin x \cos y, \sin x \sin y)$ induces an embedding of the Klein bottle in $\mathbb{E}^4$.

Proof If $(x, y), (x', y') \in [0, 2\pi] \times [0, \pi]$ satisfy given equations, then one can obtain that $(2 + \cos x)^2 = (2 + \cos x')^2$ by squaring and adding the two equations. Therefore, $\cos x = \cos x'$, so that two rest equations hold. Moreover, $y = y'$ or

$y = y' + k\pi$ for some $k \in \mathbb{Z}$. Then

$$(0, y) \sim_g (2\pi, y), \quad (x, 0) \sim_g (2\pi - x, \pi),$$

since $\sin x \cos y = \sin x' \cos y'$ and $\sin x \sin y = \sin x' \sin y'$ together with $\cos x = \cos x'$ induce $x = x'$ if $x \neq 0, 2\pi$.

So the induced map

$$\tilde{g} : ([0, 2\pi] \times [0, \pi]) / \sim_g \rightarrow \mathbb{E}^4, \quad [(x, y)] \mapsto g(x, y)$$

is well-defined and one-one. Since $[0, 2\pi] \times [0, \pi]$ is compact (so is $([0, 2\pi] \times [0, \pi]) / \sim_g$) and $\mathbb{E}^4$ is Hausdorff, $\tilde{g}$ is an embedding by the result of Exercise 3.14.

Chapter 5 Knowledge

5.1 Compactness

Proposition 5.1 (Compactness in Topological Spaces)

Let X be a topological space and $K \subset X$ be a subset. The following statements are equivalent:

- (1) K is compact.
- (2) Every open cover of K has a finite subcover.
- (3) Every collection of closed subsets of X with the FIP that intersects K has a non-empty intersection with K . ♠